**General Consequences of the Postulates**

Now let’s examine some of the general consequences of these postulates before we start analyzing any specific systems.

**Time development of the statistics of an observable**

We’ll use the Schrodinger picture here….So the Schrodinger equation tells us how wavefunction ψ evolves with time. Since the expectation of an observable depends on ψ, we can determine how the expectation of an observable depends on time. Consider the expectation of an observable, <A>, and let’s see how it evolves with time. We’ll allow for the fact that the operator may explicitly depend on time. For instance, it could be a time-dependent potential operator V(x,t) = (1/2)kx2sin(t) or something. So then:



To evaluate the first derivative, we’ll take the adjoint of the Schrodinger equation. Note,



So filling this in we get:



and so we have:



[Obviously it would’ve been easier to start with Heisenberg picture here] There is actually a lot of classical physics in this equation. The statistical average of an observable often represents the classical behavior. For instance, consider the electron diffraction experiment. Though the electrons hit almost every point on the screen. On average they cluster around the center, which is where we would expect them to be classically. So if the average of an observable roughly represents the classical expectation, we would expect the time development of the average to follow classical laws. Let’s see if this is true.

For example, we’ll consider a free particle, in one dimension. In that case, the Hamiltonian is given by:



Now let’s see how the average momentum will develop in time according to Ehrenfest’s theorem. We have:



So the average momentum (and therefore the average velocity) will remain constant throughout the motion. This is precisely what we’d expect from classical mechanics. In the absence of forces, we should have the velocity remain constant. Let’s add a force to the Hamiltonian. The force will enter in through the presence of a potential energy operator, , such that  - still working in 1D. Alright so then,



and the rate of change of <px> will be:



and so we have:



And we see that quantum mechanics has reduced to classical mechanics, on average.

So this is another encouraging indication that our formulation of quantum mechanics is on the right track. Should note, though, that this doesn’t mean our customary classical dynamics is what our quantum dynamics will reduce to. The difference lies in the expectation brackets. For instance, consider the ground state of the hydrogen atom. The classically expected behavior is of a particle orbiting the nucleus. But that’s not what this equation predicts. Since the ground state is spherically symmetric, the average force on the particle would be zero. And so indeed we should have the rate of change of momentum is zero. And that’s what we have in the ground state, or any stationary state.

**Virial Theorem**

Let’s go back to:



and let = ∙ ( is position operator, not a unit vector). Then for energy eigenstates, we have:



Let H = p2/2m + V(r). Then (implicit summation over repeated indices),



And so we have:



The harmonic oscillator and Coulomb potential are special cases of interest (T = kinetic energy, V = potential energy).



**Energy-time uncertainty relation**

Going back to Ehrenfest’s theorem,



We’ll note that if Â commutes with Ĥ then the expectation of A is constant with respect to time – provided Â itself doesn’t explicitly depend on time. A corollary is that the expectation of the energy is always constant with time (a statement of the principle of conservation of energy) since Ĥ commutes with itself. Another statement we can make is that if the state vector is itself an eigenvector of the Hamiltonian, then the expectation of A will be constant, regardless of whether Â commutes with Ĥ or not. This is because,



Another point of interest is that if A commutes with H, and |ψ> starts off in an eigenstate |ψA> of A, with eigenvalue *a*, then it will remain in a state with that expectation.



Now as we argued in that Commuting operators file, if [A,B] = 0, then [A,f(B)] = 0. And so since [A,H] = 0, we have [A,U] = 0. So we can say,



So that shows us |ψA(t)> will remain an eigenstate of the operator , with eigenvalue *a*. But do note that |ψA(t)> can be *any* eigenstate with that eigenvalue. So recapitulating:



where all the expectations are understood to be with respect to the state |ψ>. OK, now suppose though that Â doesn’t commute with Ĥ, and that |ψ> is not an eigenstate of Ĥ. And we’ll suppose that |ψ> isn’t an eigenstate of A. Then presently |ψ> will have some average value <A>, and <E>, as well as some uncertainty ΔA and ΔE. And since |ψ> isn’t an eigenstate of H, these expectations will change with time. Let’s figure out how long it will take for the expectation of A to pass through one present standard deviation from the mean, ΔA. According to the Heisenberg uncertainty principle we have:



Assuming Â doesn’t explicitly depend on time, Ehrenfest’s principle will enable us to say:



(I would say that this implicitly requires we’re evaluating the state at a certain instant in time, like t = 0), since (1/2)|<[A,H]>| assumed a given state |ψ> and (1/2)|<iℏ∂A/∂t>| is assuming a time developed state |ψ(t)>) Let Δt be the time it takes <Â> to pass through one standard deviation. This would be:



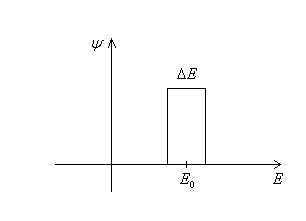
and so then we can write the relationship above as:



This expression tells us that it is the uncertainty in the energy that controls how quickly the state decays from its initial configuration. If the ΔE = 0, i.e., the state is an eigenstate of Ĥ, then all observable’s expectations remain constant. But if the state isn’t an eigenstate of Ĥ, then all observable’s expectations will change, and the larger the uncertainty in E, the quicker it will change. But now we shouldn’t interpret this decay as necessarily decoherence, it’s just some unspecified change of form.

**Energy-time uncertainty relation**

Suppose we have an energy wavefunction ψ(E) with energy uncertainty ΔE. Say something like a box of width ΔE centered about E­0.



What is its lifetime? For this we must calculate ψ(t). Well,



So we have,



and we see that this is a decaying (oscillating) function of t. Now what do we mean by ‘lifetime’ of the state? We would mean the amount of time it takes for the wavefunction to lose its character of being peaked, box-like, in energy space. Note that at t = 0, the coefficient of exp(-iE0t) is 1. As t progresses, it will become smaller. Eventually it will go to 0 (and then back up again, and back down, etc.)



We can say that it loses its character by the time it first touches down at 0. This would be after a time Δt such that



So again, we have here,



**Cyclic Behavior**

Might note that our time-development formulas suggest periodic behavior. Go back to:



And suppose the eigenvalue spectrum to be discrete, as it should be for any finite volume, no matter how large. And let Egcf be the greatest common factor of all the energy levels. Then each state would have an energy equal to Egcfnstate, where nstate = Estate/Egcf. So we could write:



And so when t = 2πℏ/Egcf, the state will be:



meaning, it has repeated itself. For instance, in 1D, where En = E0n2, and so every T = 2πℏ/E0 the motion will repeat itself.